Washington Math Olympiad

Solutions Manual

May 30, 2024

The following document contains an answer key for the short-answer section and solutions to all 20 problems.

1 Answer Key

1. 16
2. 15
3. 35
4. 2
5. 102
6. 5
7. 280
8. 96
9. 10
10. 3
11. 16
12. 23
13. 91
14. 14
15. 822
16. 31
17. 15
18. 947
19. 588
20. See solutions

2 Solutions

1. Find the number of factors of the smallest positive integer n such that $n \cdot 20!$ is a perfect square.

   Ryan Chen

   Solution. It is easy to find that $20! = 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$, so $n = 11 \cdot 13 \cdot 17 \cdot 19$, and the number of factors of n is $\boxed{16}$. □

2. Alex has 20 green marbles, 15 blue marbles, and 10 red marbles. He randomly places each of his marbles into one of three boxes colored red, blue, and green with a $\frac{1}{3}$ probability of placing each marble into any of the 3 boxes. What is the expected number of balls whose color matches the color of the box that it is inside?

   Bruce Shu

   Solution. Each marble has a $\frac{1}{3}$ chance of going in the right box. By linearity of expectation, this means that there will be $45 \cdot \frac{1}{3} = \boxed{15}$ marbles that go in the right box. □

3. A "fun coin", is a coin such that the probability to land on heads, tails, or the side is all equal. If Biff flips the coin 6 times, the probability that at some point after the first flip, Biff has flipped the coin an equal amount on its head, tail, and side can be expressed as $\frac{a}{b}$, where a and b are relatively prime. Find $a + b$.

   James Yang

   Solution. Let x, y, and z be the amount of flips that land on heads, tails, and on the side, respectively. The only possible combinations after flipping the coin 6 times are $x = y = z = 1$, or $x = y = z = 2$. Let's start with the first case, where everything is flipped once. This can only happen after 3 coin flips and can happen in $3! = 6$ ways. Thus, this probability is $\frac{6}{27} = \frac{2}{9}$. Next, let us consider the second case. Note that this can happen in $\frac{6!}{2!2!2!} = 90$ ways–similarly to rearranging the letters of a word. Our probability in this case is $\frac{90}{3^6} = \frac{10}{81}$. However, we must subtract the ways where both cases happen. This is counted by $\frac{2}{9} \cdot \frac{2}{9}$. Therefore, our answer is $\frac{2}{9} + \frac{10}{81} - \frac{4}{81} = \frac{24}{81} = \frac{8}{27}$, which gives the final answer of $\boxed{35}$. □

4. Let $f(x) = x^2 - 24x + 46$ and $g(x) = f(f(f(f(x))))$. Find $g(20) + g(2) - g(4)$.

   Laura Wang

   Solution. Note that $g(20) = f(f(f(f(20)))) = f(f(f(-34)))$ and that $g(4) = f(f(f(4))) = f(f(f(-34)))$ and that $g(2) = f(f(f(f(2)))) = f(f(f(2))) = f(f(2)) = f(2) = 2$. As a result, $g(20) + g(2) - g(4) = f(f(f(-34))) + 2 - f(f(f(-34))) = \boxed{2}$. □

5. A circle with radius 2 is in the center of a rectangle with dimensions 10 by 20. The circle rotates at a speed of one revolution per second. A flashlight is on the edge of the circle. Every $\frac{1}{16}$ of a second, the flashlight toggles between on and off. Given that the flashlight starts off facing left and is on, the total area of the region that experiences light can be expressed as $a - b\pi$. Find $a + b$.

   Ryan Tang

   Solution. Note that besides the circle in the middle, the on area is the same as the off area. Thus the answer is $\frac{200-4\pi}{2} = 100 - 2\pi$, and our answer is $100 + 2 = \boxed{102}$. □

6. Real numbers a, b, c satisfy $a^2 - b^2 = 1 + \sqrt{2}$, $b^2 - c^2 = 1 - \sqrt{2}$, and $c + a = \sqrt{2} + \sqrt{5}$. Find the value of $a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2$.

   Kaiyuan Mao

   Solution. Because $a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 = \frac{1}{2}((a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2)$ and $a^2 - c^2 = (a^2 - b^2) + (b^2 - c^2) = (1 + \sqrt{2}) + (1 - \sqrt{2})$, the value of the expression is $\frac{1}{2}((1 + \sqrt{2})^2 + (1 - \sqrt{2})^2 + 2^2) = \boxed{5}$. □

7. Fourteen circles are drawn, with 12 of the congruent circles externally tangent to the 13th circle, which is inside the twelve other circles. The 14th circle is internally tangent to the 12 circles, "encapsulating" the 13 other circles. If the radius of each of the 12 congruent circles is 34, the sum of the diameters of the other 2 circles can be expressed as $a\sqrt{b} + c\sqrt{d}$, where b and d are square-free. Find $a + b + c + d$.

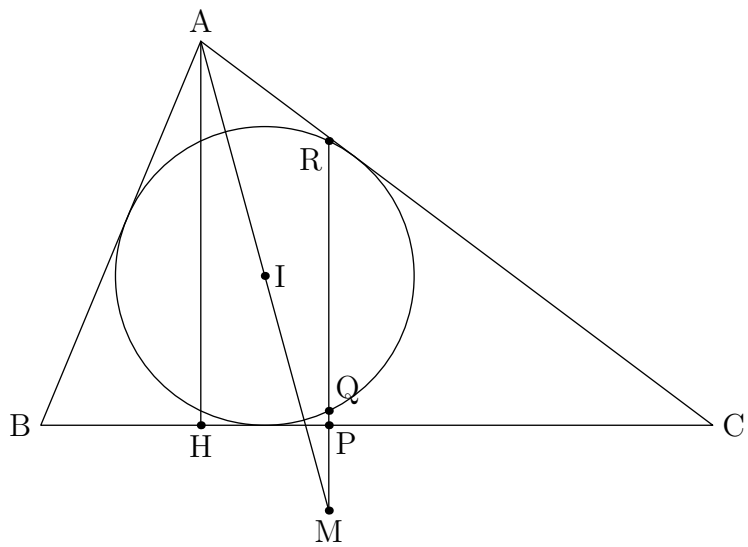
   Alexander Yang

   Solution. Connecting the center of the internal circle to the centers of the 12 encapsulating circles, we see that each triangle is a $68, 34 + r, 34 + r$ triangle with angles $30, 75, 75$, where r is the internal radii. By dropping the altitude from the center of circle with radius r, we find a $15 - 75 - 90$ triangle with hypotenuse $r + 34$ and small leg 34. Using trigonometry, we can pretty easily find that $r = 34 \cdot (\sqrt{6} + \sqrt{2} - 1)$. The larger radius is just more than the smaller by twice the radius of the small circles, or $34 \cdot (\sqrt{6} + \sqrt{2} + 1)$. Adding up the two gives us $34 \cdot 2(\sqrt{6} + \sqrt{2})$. Note that we want to find the sum of the diameters of the two circles. Thus, we have $136\sqrt{6} + 136\sqrt{2}$, so our answer is $136 + 6 + 136 + 2 = \boxed{280}$. □

8. Acute triangle ABC with incenter I and $AB = 13, AC = 20$, and $BC = 21$. Line AI is doubled in length past I to point M. The perpendicular from M to BC intersects BC at P and the incircle at Q and R, with Q on PR. Find $MP \cdot MQ \cdot MR$.

Wesley Wu

Solution.



Draw the altitude from A to BC, as shown. From power of a point, $MQ \cdot MR = MI^2 - r^2 = AI^2 - r^2$, where r is the inradius. This is also equal to the length of the tangent from A to the incircle squared. The semiperimeter of the triangle is 27, and the length of the tangent from A to the incircle is $27 - BC = 6$. Thus, $MQ \cdot MR = 36$.
Now, we must find MP. The vertical distance from A to I is equal to $AH - r$. The height is 12 and the inradius is $\frac{14}{3}$. Thus, the vertical distance from M to I is the same as the vertical distance from A to I, which is $\frac{22}{3}$. Then, $PM = \frac{22}{3} - \frac{14}{3} = \frac{8}{3}$. Thus, the product $MP \cdot MQ \cdot MR = 36 \cdot \frac{8}{3} = \boxed{96}$. □

9. Rectangle $WAMO$ has $WA = 34$ and $AM = 14$. Square $RDFE$ is drawn with RD on WA such that the combined area of square $RDFE$ and trapezoid $EFMO$ is minimized. The side length of $RDFE$ can be expressed as $\frac{a}{b}$. Find $a + b$.

   Patrick Sun

   Solution. Let the side length of square $RDFE$ be s. Then, the area of the trapezoid is $\frac{(34+s)(14-s)}{2}$, and the combined area we want to minimize is $\frac{(34+s)(14-s)}{2} + s^2 = \frac{1}{2}s^2 - 10s + 34 \cdot 14$. This has its minimum at $-\frac{b}{2a} = \frac{10}{1} = \boxed{10}$. □

10. Given that $24 \log_a x \log_b x \log_c x \log_d x = \log_a x \log_c x + \log_a x \log_d x + \log_b x \log_c x + \log_b x \log_d x$, $abcd = x^{14}$, and the minimum $a+b+c+d$ is equal to 1464, compute x.

    Wesley Wu

Solution. Divide both sides by $\log_a x \log_b x \log_c x \log_d x$ to get

$$24 = \log_x a \log_x c + \log_x a \log_x d + \log_x b \log_x c + \log_x b \log_x d$$

$$= (\log_x a + \log_x b)(\log_x c + \log_x d) = \log_x(ab)\log_x(cd).$$

Let $ab = \frac{x^7}{x^k} = x^{7-k}$ and $cd = x^7 \cdot x^k = x^{7+k}$. Then, $(7-k)(7+k) = 24$ and $k = \pm 5$. Thus, the minimum of $a+b+c+d$ is equal to $2\sqrt{x^2}+2\sqrt{x^{12}}$ by AM-GM, so $x + x^6 = 732$. Therefore, $x = \boxed{3}$. □

11. Let S be the sum of the squares of the factors of 1434. Compute the sum of the digits of $\sqrt{S}$.

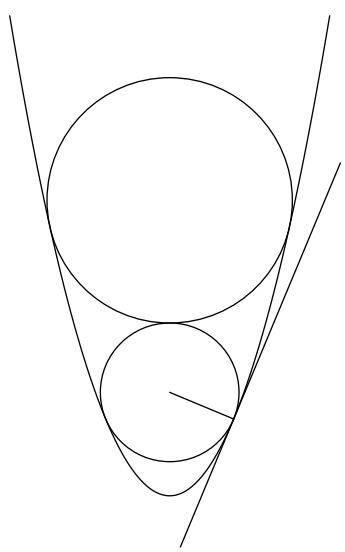
Ryan Tang

Solution. $S = (1+2^2)(1+3^2)(1+239^2) = 5 \cdot 10 \cdot (1+239^2)$. Note that since S is a perfect square, $100 \cdot \frac{1+239^2}{2}$ is a perfect square, so $\frac{1+239^2}{2}$ is a perfect square. We know that this number ends in 22, so the units digit of the number is 1, so the units digit of the square root is 1 or 9. Also, we know that this number is close to $\frac{1+240^2}{2} \approx \frac{240^2}{2} = 28800$. This is really close to 169^2. So $S = 100 \cdot 169^2$ and $\sqrt{S} = 1690$. The sum of the digits is $\boxed{16}$. □

12. The parabola $y = \frac{1}{10}x^2$ is drawn, and a circle with radius 13 is drawn tangent to the parabola at two points. Another larger circle is drawn tangent to the parabola at two points and the first circle. What is the radius of the larger circle?

Wesley Wu

Solution.



Let's ignore the large circle for a moment. The slope of the tangent line to our parabola at $x = k$ is $2k \cdot \frac{1}{10} = \frac{k}{5}$. Thus, the slope of the radius pictured in the diagram is $-\frac{5}{k}$. That radius passes through point $\left(k, \frac{k^2}{10}\right)$, so its equation is

$$y - \frac{k^2}{10} = -\frac{5}{k}(x - k).$$

Since the circle must be centered, the center is $\left(0, \frac{k^2}{10} + 5\right)$. From the distance formula, we find that the distance between the two points is $\sqrt{k^2 + 25}$.
Let k be the x-coordinate of the intersection of the radius and the parabola. We know $\sqrt{k^2 + 25} = 13$ and $k = 12$. Thus, the center of the smaller circle is $\left(0, \frac{97}{5}\right)$.
Let's bring back the big circle. We can use the same formulas. First, define n be the x-coordinate of the rightmost tangent point of the circle to the parabola; also define r as its radius. The coordinate of the center of the big circle is $\frac{97}{5} + 13 + r = \frac{162}{5} + r = \frac{n^2}{10} + 5$. In addition, from the distance formula, $n^2 = k^2 + 25$. Therefore, we have the system

$$\begin{cases} n^2 = 10r + 274 \\ n^2 = \quad r^2 - 25 \end{cases}$$

which by substitution becomes $r^2 - 10r - 299 = 0$ which has the positive solution $\boxed{23}$. □

13. Ryan has a list of the first 100 positive square numbers. He then randomly shuffles the list and concatenates all the numbers together in that order. The expected remainder of this number when divided by 90 can be expressed as $\frac{a}{b}$, where a and b are coprime. Find $a + b$.

Ryan Tang

Solution. Note that the remainder of the concatenation is constant modulo 9– namely, it is equal to $\frac{100 \cdot 101 \cdot 201}{6} \equiv 50 \cdot 101 \cdot 67 \equiv 5 \cdot 2 \cdot 4 \equiv 4 \pmod 9$. Next, we must find the possible remainders modulo 10. This is equivalent to finding the possible last digits of the concatenated number. Of the 100 possible square numbers that are at the end of the number, 10 end in 0, 20 end in 1, 20 end in 4, 10 end in 5, 20 end in 6, and 20 end in 9. Using CRT, the ending digits modulo 90 become 40, 31, 4, 85, 76, and 49, respectively. Taking the weighted expected value, our answer is $\frac{40+31\cdot 2+4\cdot 2+85+76\cdot 2+49\cdot 2}{10} = \frac{445}{10} = \frac{89}{2}$, giving us the answer $\boxed{91}$. □

14. How many integers x satisfy $0 \le x \le 314$ and $\sin(x)\sin(x + 3) > 0$?

Ryan Tang

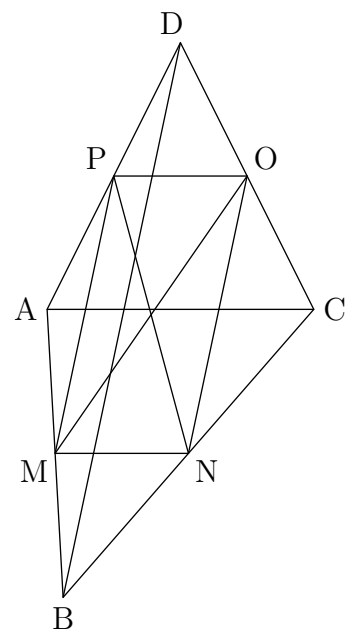
Solution. Note that the distance between two consecutive roots can have either three or four integers between them. The ones with four integers between them

are the ones that satisfy the given inequality. There are a total of 100 of these gaps (because $0 \le x \le 314$). Thus, we have $a+b=100$ and $3a+4b=314$, where a represents the number of gaps with 3 integers and b represents the number of gaps with 4 integers. Solving for a and b, we get $b = \boxed{14}$. $\square$

15. In quadrilateral $ABCD$, the midpoints of AB, BC, CD, and DA are M, N, O, and P, respectively. Furthermore, let the midpoint of MN be Q. Given that $MO = 4\sqrt{10}$, $NP = 2\sqrt{29}$, $BD = 2\sqrt{113}$, $DQ = 11\sqrt{2}$, and $AD = CD$, the perimeter of $ABCD$ can be expressed as $\sqrt{a}+\sqrt{b}+\sqrt{c}$. Find $a+b+c$.

Wesley Wu

Solution.



By Varignon's theorem, $MNOP$ is a parallelogram. Since MP is a midsegment of triangle ABD, $2MP = BD$ and $MP = NO = \sqrt{113}$. Also, we know $MO = 4\sqrt{10}$ and $NP = 2\sqrt{29}$. Let the intersection of NP and MO be K. We can find $\cos(MKO)$ using law of cosines:

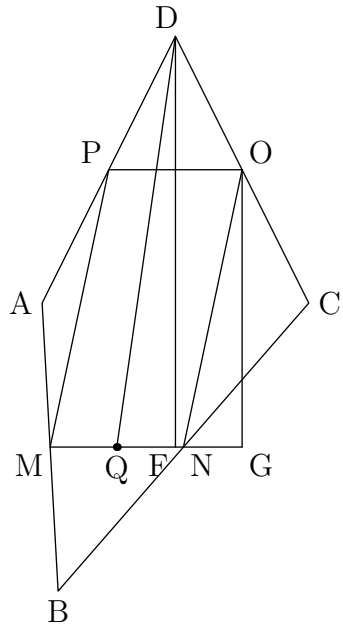
$$PK^2+MK^2-2PK\cdot MK\cos(MKO) = MP^2 \to 69-2\cos(MKO)\sqrt{29\cdot 40} = 113.$$

Thus, $\cos(MKO) = -\frac{22}{2\sqrt{290}} = -\frac{11}{\sqrt{290}}$. Using law of cosines on PKO, we find

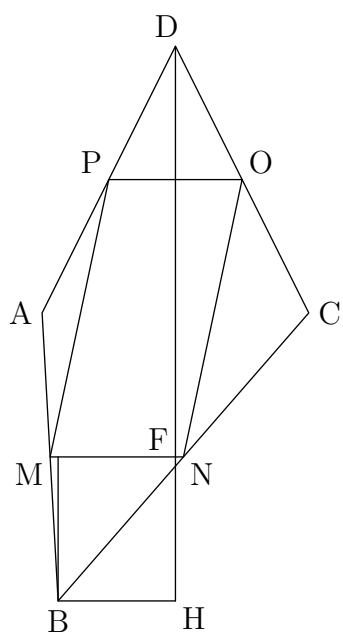
$$PO^2 = 69 - 4\sqrt{290}\cos(PKO) = 69 + 4\sqrt{290}\cos(MKO) = 69 - 4\cdot 11 = 25.$$

Thus, $OP = MN = 5$ and $AC = 10$. Now, let's quickly find the area of $MNOP$ using the product of the diagonals and sine formula:

$$[MNOP] = \frac{1}{2}\cdot 4\sqrt{10}\cdot 2\sqrt{29}\cdot\sin(PKO) = 4\sqrt{290}\cdot\frac{13}{\sqrt{290}} = 52.$$



Now, we drop the perpendicular from D to MN and the perpendicular from O to MN. We can find OG using $\frac{[MNOP]}{MN} = \frac{52}{5}$ and $NG = \frac{11}{5}$. Also, we know $FQ = QN + NG - FG = \frac{5}{2} + \frac{11}{5} - \frac{5}{2} = \frac{11}{5}$. Therefore, $DF = \sqrt{242 - \frac{121}{25}} = \frac{77}{5}$. Thus, the height of isosceles triangle OPD is 5. Using Pythagorean theorem, we find $CD = AD = 5\sqrt{5}$.



Extend DF past F to H so that BH is parallel to MN. From Pythagorean theorem, $BH^2 = BD^2 - DH^2 = 452 - \left(\frac{104}{5}\right)^2 = \frac{484}{25}$ and $BH = \frac{22}{5}$. Also,

$FN = \frac{5}{2} - \frac{11}{5} = \frac{3}{10}$.
Since the height of the parallelogram is $\frac{52}{5}$, and the height of trapezoid $APOC$ is 5. Thus, the height of trapezoid $AMNC$ is $\frac{27}{5}$ and so is FH.
Using the Pythagorean theorem again, we find

$$BN^2 = \left(\frac{3}{10} + \frac{22}{5}\right)^2 + \left(\frac{27}{5}\right)^2.$$

Thus, $BN = \frac{\sqrt{205}}{2}$ and $BC = \sqrt{205}$.
Let the base of the perpendicular from B to MN be J. We know $MJ = MN - JN = \frac{3}{10}$. Thus, using Pythagoras one last time, we find $MB = \sqrt{\left(\frac{3}{10}\right)^2 + \left(\frac{27}{5}\right)^2} = \frac{\sqrt{117}}{2}$. Thus, $MB = \sqrt{117}$.
Adding all of these, we find that the perimeter of $ABCD$ is $\sqrt{117} + \sqrt{205} + \sqrt{500}$.
Our desired answer is $117 + 205 + 500 = \boxed{822}$. □
(Note: A lattice point quadrilateral with coordinates $(0, 0), (6, 9), (16, 14)$, and $(14, 3)$ satisfies all these conditions.)

16. Uh oh! While writing out Pascal's Triangle, Yaptrick forgot the recursive property for the triangle, and instead of adding the two numbers above a space to create a new number, he added an extra 1 to the sum. The first few rows of his triangle look like this:

1	(Row 0)
2 2	(Row 1)
3 5 3	(Row 2)
4 9 9 4	(Row 3)

He defined $\left[{m \atop n}\right]$ as the $(n+1)^{\text{th}}$ number in Row m of his triangle. For example, $\left[{3 \atop 2}\right] = 9$ and $\left[{2 \atop 1}\right] = 5$. Find the remainder when $\left[{20 \atop 10}\right]$ is divided by 100.

Ryan Chen

Solution. Note that Yaptrick's triangle is the sum of many Pascal's triangles,

one for each entry. Thus,

$$\begin{aligned}
\begin{bmatrix}20\\10\end{bmatrix} &= \sum_{i=0}^{20}\sum_{j=0}^{i}\binom{20-i}{10-j}\\
&= \sum_{i=0}^{10}\sum_{j=i}^{i+10}\binom{j}{i}\\
&= \sum_{i=0}^{10}\binom{i+11}{i+1}\\
&= \sum_{i=0}^{10}\binom{i+11}{10}\\
&= \sum_{i=0}^{11}\binom{i+10}{10}-1\\
&= \binom{22}{11}-1
\end{aligned}$$

Trivially, $\binom{22}{11} \equiv 0 \pmod 4$.
Also, because $4 = \nu_5(22!) = 2\nu_5(11!)$, the 5's cancel out in the denominator and numerator, so $\frac{22!}{(11!)^2} = \frac{\frac{22!}{5^4}}{\frac{(11!)^2}{5^4}}$. We get that

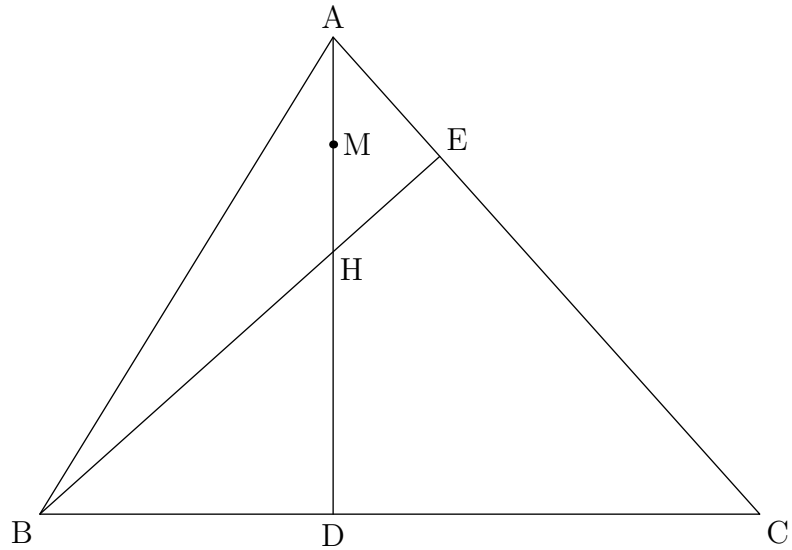
$$\begin{aligned}
1\cdot 2\cdot 3\cdot 4\cdot \tfrac{5}{5}\cdot 6\cdot 7\cdots \tfrac{20}{5}\cdot 21\cdot 22 &\equiv\\
(1\cdot 2\cdot 3\cdot 4)\cdot 1\cdot (6\cdot 7\cdot 8\cdot 9)\cdots 4\cdot -4\cdot -3 &\equiv\\
-1\cdot -1\cdot -1\cdot -1\cdot (1\cdot 2\cdot 3\cdot 4)\cdot 12 &\equiv\\
-12 &\equiv\\
13 \pmod{25}&
\end{aligned}$$

By similar reasoning $\frac{11!}{25} \equiv -3 \pmod{25} \implies \frac{(11!)^2}{5^4} \equiv 9 \pmod{25}$.

Therefore, $\binom{22}{11} \equiv \frac{13}{9} \equiv 7 \pmod{25}$. Finally, by CRT, $\binom{22}{11} \equiv 32 \pmod{100}$, so the answer is $32 - 1 = \boxed{31}$. □

17. In acute triangle ABC, the orthocenter is H. The midpoint of AH is M. If $AM^2 = \frac{9}{5}$, $\left(\frac{CM}{BM}\right)^2 = \frac{83}{58}$, and $BC = 9$, find the value of $AB + AC$.
Wesley Wu

Solution (Sebastian Prasanna):



Let $BD = a$ and $MD = b$. Using the Pythagorean theorem, we find $CM^2 = (9-a)^2 + b^2$ and $BM^2 = a^2 + b^2$. Therefore, we have

$$\frac{(9-a)^2 + b^2}{a^2 + b^2} = \frac{83}{58} \to 25a^2 + 25b^2 + 58 \cdot 18a = 81 \cdot 58.$$

We also know that AEH is similar to ADC and BDH. Thus, ADC is similar to BDH and $\frac{AD}{DC} = \frac{BD}{DH} \to \frac{b+\frac{3}{\sqrt{5}}}{9-a} = \frac{a}{b-\frac{3}{\sqrt{5}}}$. Thus, $b^2 = -a^2 + 9a + \frac{9}{5}$. Substituting this into the first equation, we get $25a^2 - 25a^2 + 25 \cdot 9a + 45 + 58 \cdot 18a = 81 \cdot 58$, which leads to

$$25 \cdot 9a + 116 \cdot 9a = 522 \cdot 9 - 5 \cdot 9 \to 141a = 517 \to a = \frac{11}{3}.$$

We can easily find that $b = \frac{31}{3\sqrt{5}}$ and $AD = \frac{8\sqrt{5}}{3}$, and using the Pythagorean theorem, we find $AB = 7$ and $AC = 8$, so the desired answer is $\boxed{15}$. □

18. Find the remainder when the sum of the digits of $\lfloor \frac{10^{10^{100}}}{1001} \rfloor$ is divided by 1001.

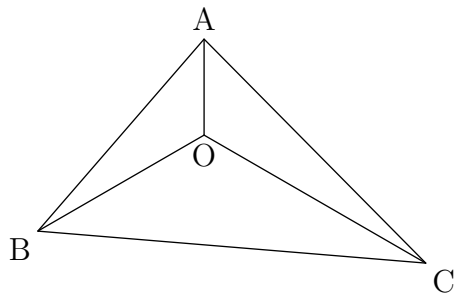
Ryan Tang

Solution. Note that $\frac{1}{1001} = \frac{999}{999999} = 0.\overline{000999}\ldots$. Thus, $\lfloor \frac{10^{10^{100}}}{1001} \rfloor = 999000\ldots9990009$, where there are $10^{100} - 3$ digits, since $10^{100} - 3 \equiv 1 \pmod 6$. The number of 9's in this number is equal to $\frac{10^{100}-4}{2} + 1$, since besides the 9 at the end, there are an equal number of 0's and 9's. Take mod 1001 to get that $\frac{10^{100}-4}{2} + 1 \equiv 5 \cdot 10^{99} - 1 \equiv 5 \cdot (-1)^{33} - 1 \equiv 995 \pmod{1001}$. Thus, the sum of the digits equals $995 \cdot 9 \equiv \boxed{947} \pmod{1001}$. □

19. Let a, b, and c be the roots of $x^3 - 11x^2 + 14x - 2$. Let $x = \sqrt{a^2 + b^2 + ab}$, $y = \sqrt{b^2 + c^2 + bc}$, and $z = \sqrt{c^2 + a^2 + ca}$. Compute $(x + y + z)(-x + y + z)(x - y + z)(x + y - z)$.

Ryan Tang

Solution.



Note that this is equal to $4 \cdot$ (area of a triangle with side lengths $x, y, z)^2$. Using law of cosines, we notice that x, y, z represent the side lengths of a triangle with $AO = a$, $BO = b$, and $CO = c$, where O is the point such that $\angle AOB = \angle BOC = \angle COA = 120°$. Then, using law of sines, we find that the area of this triangle is equal to $(ab + bc + ac) \cdot \frac{\sin 120°}{2}$. This is equal to $7\sqrt{3}$ by Vieta's, and the answer is $4 \cdot (7\sqrt{3})^2 = \boxed{588}$. □

20. Let f be a monic quadratic with integer coefficients of the form $f(x) = x^2 - ax + b$ with $a, b > 0$. With proof, find all functions f such that all the roots of $f(f(x))$ are integers.

Wesley Wu

Solution. Let $f(x)$ not be a square. If $f(f(x))$ has integer roots, say a, b, c, d, and WLOG $f(a) = f(b) =$ first root of f(x) and $f(c) = f(d) =$ second root of f(x), then $f(a)$ and $f(c)$ are both integers as $f(x)$ is an integer polynomial. Therefore, if $f(f(x))$ has integer roots, $f(x)$ must too. Thus, let $f(x) = x^2 - (p + q)x + pq$ with roots p and q (WLOG $p > q$). The roots of $f(f(x))$ are the same as the roots of $x^2 - (p + q)x + pq - p$ and $x^2 - (p + q)x + pq - q$. By their discriminants, $(p + q)^2 - 4pq + 4p = (p - q)^2 + 4p$ is a square and so is $(p - q)^2 + 4q$.

Let $k = p - q$ or $p = k + q$. Then, $k^2 + 4q$ and $k^2 + 4k + 4q$ are both squares. Let $k^2 + 4q = a^2$ and $k^2 + 4k + 4q = b^2$. Then, $b^2 - a^2 = (b - a)(b + a) = 4k$. Since the right hand side is even, $b - a$ and $b + a$ must both be even. Thus, we can let $k = mn$ where m and n are positive integer factors. Then, we can split the 4 into $2 \cdot 2$ and distribute it to the left hand side. We have $b - a = 2m$ and $b + a = 2n$.

Substituting this back into $k^2 + 4q = a^2$, we get $(mn)^2 + 4q = (n - m)^2 \to (n - m)^2 - (mn)^2 = 4q \to (n - m - mn)(n - m + mn) = 4q$. Since $b - a < b + a$,

we have $m < n$, so $n+mn > m$ and $n-m+mn > 0$. For the other factor, we have $n-m-mn = -(mn+m-n) = -((m-1)(n+1)+1)$. If $(m-1)(n+1)$ is negative, then either $n < -1$ or $m < 1$. However, m and n are positive integer factors of k, which contradicts the previous statement, so $n - m - mn < 0$. Therefore, $(n-m-mn)(n-m+mn) < 0$, but $4q > 0$, which is a contradiction. Thus, $f(x)$ must be a square.

Now, let $f(x)$ be a square. Then, $f(x) = x^2-2px+p^2$, and the roots of $f(f(x))$ are the solutions to the equation $x^2-2px+p^2 = p$. By discriminant, $(2p)^2-4(p^2-p) = 4p$ is a square. Then, p must be a square number. Thus, the functions

$$\boxed{f(x) = x^2 - 2k^2x + k^4, k \in \mathbb{N}}$$

are the only ones that work. $\square$